QUARTIC SURFACES INVARIANT UNDER PERIODIC TRANSFORMATIONS.

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In 1845 Steiner* stated the following theorem: "Let P and Q be two fixed points on a plane cubic curve (or double points on a plane quartic curve) and A a variable point on the curve. Let PA meet the curve again in A_1, QA_1 in A_2, PA_2 in A_3, $\cdots$, QA_{2n-1} in A_{2n}. If A_{2n} coincides with A for one position of A, then it coincides with A for every position of A." In 1910 Snyder† considered a quartic surface having two conical points P and Q, and stated the condition that the two transformations A into A_1 and A_1 into A_2 should be commutative for the section of the quartic surface by any plane through the line PQ. The double transformation A into A_2 is then of period two. That is if S is the first transformation and T the second, then $(ST)^2 = 1$. This suggested to the late Professor J. E. Wright, of Bryn Mawr, the problem of finding quartic surfaces such that $(ST)^3 = 1$. His untimely death prevented him from solving the problem. Professor Snyder, of Cornell, recently proposed it to us, and the solution is given in this paper. It may also be interpreted as the condition that the two involutorial transformations S and T of the general (2, 2) correspondence satisfy the condition $(ST)^3 = 1$.

The above theorem of Steiner follows easily from the expression of the coördinates of any point on a non-singular cubic curve in terms of elliptic functions $p(u)$ of a parameter u

$$x_1 = \rho p'(u), \qquad x_2 = \rho p(u), \qquad x_3 = \rho.$$

It is well known that the coördinates can be so chosen that the sum of the parameters of three collinear points on the curve is equal to a sum of the multiples of the periods $2\omega_1$, $2\omega_2$.

Denoting the parameter of a point by the corresponding small letter, we have (mod $2\omega_1$, $2\omega_2$)

$$p + a + a_1 \equiv 0,$$

$$q + a_1 + a_2 \equiv 0.$$

Therefore

$$p - q + a - a_2 \equiv 0.$$

* Crelle, vol. 32 (1845), pp. 182–184.

† Trans. Am. Math. Soc., vol. 11, p. 16, Sturm, Geo., Verwandtschaften, Band I, p. 267.

Similarly

$$p - q + a_2 - a_4 \equiv 0$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$p - q + a_{2n-2} - a_{2n} \equiv 0.$$

Hence if A_{2n} coincides with A, by addition we find

$$n(p - q) \equiv 0$$

which is independent of the position of A. The parameters of P and Q are seen to differ by one nth of a period.

If we invert with respect to a triangle PQR, where R is a point not on the cubic, the lines through P and Q are inverted into lines through P and Q, but the cubic is inverted into a quartic having P and Q for double points, so the theorem holds in the latter case.

The condition for periodicity may be expressed in a simple geometric form by taking the limiting case of the theorem as A approaches P. For a cubic curve and period two, A_3 is the point where the line PQ again meets the curve. Also PA and QA_2 are the tangents at the points P and Q respectively. The condition for per od two is therefore that these tangents meet on the curve at the point A_1. For period three A_5 is the point where PQ again meets the curve and PA and QA_4 are the tangents at P and Q respectively. If these tangents meet the curve again in A_1 and A_3, then the condition is that the lines QA_1 and PA_3 meet on the curve in the point A_2.*

For a quartic curve with double points P and Q and period two, PA and PA_3 are the tangents at P, also A_1 and A_2 are their points of intersection with the curve. The condition therefore is that the points QA_1A_2 are collinear.

For period three the tangents PA and PA_5 at P meet the curve in two points A_1, A_4 such that QA_1 and QA_4 meet the curve in two points A_2 and A_3 which are collinear with P.

We will now proceed to express these conditions analytically. Using homogeneous coördinates $x_1x_2x_3x_4$, the equation of a quartic surface having conical points at

$$P = (0, 0, 0, 1) \quad \text{and} \quad Q = (0, 0, 1, 0)$$

is of the form

$$(1) \quad (a_1x_3^2 + b_1x_3 + c_1)x_4^2 + (a_2x_3^2 + b_2x_3 + c_2)x_4 + (a_3x_3^2 + b_3x_3 + c_3) = 0$$

or

$$(2) \quad (a_1x_4^2 + a_2x_4 + a_3)x_3^2 + (b_1x_4^2 + b_2x_4 + b_3)x_3 + (c_1x_4^2 + c_2x_4 + c_3) = 0,$$

where the coefficients are homogeneous functions of x_1 and x_2 such that the equations are homogeneous and of degree four in the coördinates.

* Crelle, vol. 32, pp. 182–184.

The form (1) shows that the transformation S interchanges the points (x_1, x_2, x_3, x_4) and (x_1, x_2, x_3, x_4') where x_4 and x_4' are the roots of the quadratic (1) in x_4. The form (2) similarly shows that the transformation T interchanges the points (x_1, x_2, x_3, x_4) and (x_1, x_2, x_3', x_4) where x_3 and x_3' are the roots of the quadratic (2) in x_3.

If we keep x_1/x_2 fixed, we have the section of the quartic surface by a plane through P and Q. This section has double points at P and Q and has for tangents at P

$$a_1x_3^2 + b_1x_3 + c_1 = 0. \tag{3}$$

First the analytic condition for period two will be deduced. Denoting the roots of (3) by x_3, x_3' the tangents at P meet the surface at

$$A_1 = (x_1, x_2, x_3, x_4)$$

and

$$A_2 = (x_1, x_2, x_3', x_4')$$

where from (1)

$$x_4 = -\frac{a_3x_3^2 + b_3x_3 + c_3}{a_2x_3^2 + b_2x_2 + c_2}, \qquad x_4' = -\frac{a_3x_3'^2 + b_3x_3' + c_3}{a_2x_3'^2 + b_2x_2' + c_2}. \tag{4}$$

Hence, by subtracting and dividing by $x_3 - x_3'$, we have

$$\frac{x_4 - x_4'}{x_3 - x_3'} = \frac{C_1x_3x_3' - B_1(x_3 + x_3') + A_1}{(a_2x_3x_3' - c_2)^2 + [b_3x_3x_3' + c_2(x_3 + x_3')][b_2 + a_2(x_3 + x_3')]}, \tag{5}$$

where the large letters denote the cofactors of the corresponding small letters in the determinant

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \tag{6}$$

But from (3)

$$a_1x_3x_3' = c_1 \qquad \text{and} \qquad a_1(x_3 + x_3') = -b_1.$$

Hence (5) becomes

$$\frac{x_4 - x_4'}{x_3 - x_3'} = \frac{a_1\Delta}{r}, \tag{7}$$

where

$$r = B_3^2 - A_3C_3.$$

If the transformation is of period two we proved that the points QA_1A_2 must be collinear. Hence $x_4 = x_4'$ and therefore

$$\Delta = 0.^* \tag{8}$$

* Sturm, Geo., Verwandtschaften, Band I, p. 267.

This is therefore the condition that the transformation be of period two for the section considered. It is of degree six in x_1 and x_2. There are therefore in general six planes through PQ which satisfy the relation $(ST)^2 = 1$. If however the seven coefficients of this equation are all zero, then all sections through PQ will satisfy the relation $(ST)^2 = 1$. The twenty-seven coefficients of (1) must therefore satisfy seven conditions in order that the surface may be invariant under a transformation ST such that $(ST)^2 = 1$.

For period three the old A_2 becomes A_4 while the new

$$A_2 = (x_1, x_2, x_3'', x_4)$$

and

$$A_3 = (x_1, x_2, x_3''', x_4').$$

From (2) follow the relations

$$(9)\qquad \begin{aligned} x_3 + x_3'' &= -\frac{b_1x_4^2 + b_2x_4 + b_3}{a_1x_4^2 + a_2x_4 + a_3}, \\ x_3' + x_3''' &= -\frac{b_1x_4'^2 + b_2x_4' + b_3}{a_1x_4'^2 + a_2x_4' + a_3}. \end{aligned}$$

But A_2, A_3 and P are collinear. Therefore $x_3'' = x_3'''$.
Hence from (9) it follows that

$$(10)\qquad \frac{x_3 - x_3'}{x_4 - x_4'} = \frac{C_3x_4x_4' - C_2(x_4 + x_4') + C_1}{(a_1x_4x_4' - a_3)^2 + [a_2x_4x_4' + d_3(x_4 + x_4')][a_2 + a_1(x_4 + x_4')]}$$

Now from (4) it can be shown that

$$x_4 + x_4' = \frac{2B_2B_3 - A_2C_3 - A_3C_2}{B_3^2 - A_3C_3}$$

and

$$x_4x_4' = \frac{B_2^2 - A_2C_2}{B_3^2 - A_3C_3}.$$

For brevity let

$$2B_2B_3 - A_2C_3 - A_3C_2 = s; \qquad B_2^2 - A_2C_2 = q;$$

$$2B_1B_3 - A_1C_3 - A_3C_1 = t; \quad 2B_1B_2 - A_1C_2 - A_2C_1 = u; \quad B_1^2 - A_2C_1 = p.$$

Then

$$x_4 + x_4' = \frac{s}{r}$$

and

$$x_4x_4' = \frac{q}{r}.$$

By substituting in (10) we have

$$\frac{x_3 - x_3'}{x_4 - x_4'} = \frac{(C_3q - C_2s + C_1r)r}{(a_1q - a_3r)^2 + (a_2q + a_3s)(a_2r + a_1s)}. \tag{11}$$

If then between (6) and (11) the ratio $\frac{x_3 - x_3'}{x_4 - x_4'}$, be eliminated, we have

$$\frac{r}{a_1\Delta} = \frac{(C_3q - C_2s + C_1r)r}{(a_1q - a_3r)^2 + (a_2q + a_3s)(a_2r + a_1s)} \tag{12}$$

and hence

$$a_1\Delta(C_3q - C_2s + C_1r) = (a_1q - a_3r)^2 + (a_2q + a_3s)(a_2r + a_1s). \tag{13}$$

Using the identity

$$a_2^2q + a_2a_3s + a_3^2r \equiv a_1^2p + a_1C_1\Delta$$

and dividing out a_1 as a factor we have

$$\Delta(C_3q - C_2s) = a_1(q^2 + pr) - 2a_3qr + a_2qs + a_3s^2.$$

Now using the identities

$$a_3s + 2a_2q \equiv - a_1u - C_2\Delta,$$

$$a_2s + 2a_3r \equiv - a_1t - C_3\Delta,$$

and dividing out a_1 as a factor, we have

$$q^2 + pr - su + qt = 0. \tag{14}$$

This is the condition that the two involutorial transformations S and T of the general (2 2) correspondence (1) satisfy $(ST)^3 = 1$.

It is of degree sixteen in x_1 and x_2. Hence there are in general sixteen sections of (1) by planes through PQ such that the condition $(ST)^3 = 1$ is satisfied. If however the twenty-seven coefficients of (1) are such that the seventeen coefficients of (14) are all zero, then all sections through PQ will satisfy the relation $(ST)^3 = 1$.

A similar method applies to period four, but the degree of the condition found is greater than twenty-six in x_1 and x_2, hence it seems doubtful that there exist quartic surfaces invariant under this type of transformation.

The condition given in (14) remains true in all cases but the proof given appears to fail when $a_1 = 0$. If we keep x_1/x_2 fixed as before, we have the section of the quartic surface by a plane through P and Q. This quartic section degenerates into a cubic and the line PQ. The tangent PA at P to the cubic, is

$$x_3 = - \frac{c_1}{b_1}. \tag{15}$$

This meets the cubic again in

$$A_1 = \left(x_1, x_2, \frac{-c_1}{b_1}, x_4\right)$$

while the line QA_1 is

$$x_4 = -\frac{a_3x_3^2 + b_3x_3 + c_3}{a_2x_3^2 + b_2x_3 + c_2}. \tag{16}$$

The other tangent at P in the general case, namely PA_4, degenerates as $a_1 \doteq 0$, into the line PQ, but in such a way that the tangent at Q to the cubic meets it again in A_3, QA_3 being

$$x_4' = \frac{-a_3}{a_2}. \tag{17}$$

If QA_1 meets the cubic in

$$A_2 \equiv (x_1, x_2, x_3'', x_4),$$

then from (1) and (2)

$$\frac{-b_1}{c_1} + \frac{1}{x_3''} = -\frac{b_1x_4^2 + b_2x_4 + b_3}{c_1x_4^2 + c_2x_4 + c_3}. \tag{18}$$

and if

$$A_3 \equiv (x_1, x_2, x_3''', x_4'),$$

then

$$\frac{1}{x_3'''} = -\frac{b_1x_4'^2 + b_2x_4' + b_3}{c_1x_4'^2 + c_2x_4' + c_3}. \tag{19}$$

But PA_2A_3 are collinear, hence $x_3'' = x_3'''$. We therefore have

$$\frac{b_1}{c_1} = \frac{(x_4 - x_4')[A_3x_4x_4' - A_2(x_4 + x_4') + A_1]}{(c_1x_4x_4' - c_3)^2 + [c_2x_4x_4' + c_3(x_4 + x_4')][c_2 + c_1(x_4 + x_4')]}. \tag{20}$$

But from (15), (16), and (17)

$$x_4 - x_4' = \frac{b_1\Delta}{r}, \tag{21}$$

where x_4 and x_4' are the roots of

$$rx_4^2 - sx_4 + q = 0. \tag{22}$$

Hence (20) becomes

$$C_1\Delta(A_3q - A_2s + A_1r) = (c_1q - c_3r)^2 + (c_2r + c_3s)(c_2r + c_1s).$$

This differs from (13) only in having a's instead of c's and therefore leads to the same result (14) as this condition is unaltered when these letters are interchanged.

This method also fails when $c_1 = 0$, but corresponding to the equations

of the last case, we have

(15′) $$x_3 = 0,$$

(16′) $$x_4 = \frac{-c_3}{c_2},$$

(17′) $$x_4' = \frac{-a_3}{a_2},$$

(18′) $$x_3'' = -\frac{b_1x_4^2 + b_2x_4 + b_3}{a_2x_4 + a_3},$$

(19′) $$\frac{1}{x_3'''} = -\frac{b_1x_4'^2 + b_2x_4' + b_3}{c_2x_4' + c_3}.$$

Hence the condition $x_3'' = x_3'''$ leads to

$$(b_1q - b_3r)^2 + (b_2q + b_3s)(b_1s + b_2r) = 0,$$

which when it is transformed as in the previous cases and the factor b_1^2 is divided out, gives as before

(14) $$q^2 + pr - su + qt = 0.$$

If $a_1 = c_1 = a_3 = 0$, then $q = 0$ and the condition (14) reduces to

$$(a_2c_2 - b_1b_3)c_3^2 - b_2b_3c_2c_3 + b_3^2c_2^2 = 0.$$

The planes $x_3 = 0$ and $x_4 = 0$ are now tangent planes at P and Q respectively.

If $c_3 = c_2x_2$, then (14) becomes

$$(a_2c_2 - b_1b_3)x_2^2 - b_2b_3x_2 + b_3^2 = 0$$

and if $c_2 = b_3$, then

$$(a_2 - b_1)x_2^2 - b_2x_2 + b_3 = 0.$$

Equation (1) now takes the form

$$b_1(x_3x_4^2+x_3x_2^2)+a_2(x_3^2x_4-x_3x_2^2)+b_2(x_3x_4+x_2x_3)+c_2(x_2+x_3+x_4)=0.$$

This appears to be the simplest type of surface that fulfils the condition (14). It contains eleven arbitrary constants.

Let K_n' denote a cone of order n with vertex at (0, 0, 0, 1) and K_n'' also a cone of order n with vertex at (0, 0, 1, 0).

Equation (1) may then be written in the form

(20) $$K_2'x_4^2 + K_3'x_4 + K_4' = 0$$

and equation (2)

(21) $$K_2''x_3^2 + K_3''x_3 + K_4'' = 0.$$

Therefore the transformation S is

$$(22)\qquad \begin{aligned} x_1 &= x_1'K_2', & x_3 &= x_3'K_2', \\ x_2 &= x_2'K_2', & x_4 &= -x_4'K_2' - K_3' \end{aligned}$$

This is a transformation of monoidal type.* The image of any plane not passing through (0, 0, 0, 1) is a cubic surface with a conical point at (0, 0, 0, 1), the image of this point being $K_2' = 0$. The fundamental curves are the six lines $K_2' = 0$, $K_3' = 0$. Similarly T is

$$(23)\qquad \begin{aligned} x_1' &= x_1''K_2'', & x_2' &= x_2''K_2'', \\ x_3' &= -x_3''K_2'' - K_3'', & x_4' &= x_4''K_2''. \end{aligned}$$

Hence ST is

$$(24)\qquad \begin{aligned} x_1 &= x_1''K_2''[a_1(x_3''K_2'' + K_3'')^2 - b_1(x_3''K_2'' + K_3'')K_2'' + c_1K_2''^2] \\ &= x_1''K_2''F_6, \\ x_2 &= x_2''K_2''F_6, \\ x_3 &= -(x_3''K_2'' + K_3'')F_6, \\ x_4 &= -(x_4''F_6 + F_7)K_2'', \end{aligned}$$

where

$$F_7 = a_2(x_3''K_2'' + K_3'')^2 - b_2(x_3''K_2'' + K_3'')K_2'' + c_2K_2''^2.$$

This is a Cremona transformation. The image of any plane not passing through (0, 0, 0, 1) nor (0, 0, 1, 0) is a surface of degree nine, having a six fold point at (0, 0, 1, 0) and a conical point at (0, 0, 0, 1). The images of these points are $F_6 = 0$ and $K_2'' = 0$ respectively.

The sum of the degrees of the fundamental curves† is $9^2 - 9 = 72$. These curves are the six lines $K_2'' = 0$, $K_3'' = 0$ counted nine times, and the six cubics into which T transforms the six lines $K_2' = 0$, $K_3' = 0$.

Hence ST transforms the plane sections of (20) into curves of degree 36 passing four times through (0, 0, 0, 1) and twelve times through (0, 0, 1, 0); similarly for TS. Since ST is of period three it follows that the first triply infinite linear system of curves is transformed by ST into the second system, which is also of degree 36, but passes twelve times through (0, 0, 0, 1) and four times through (0, 0, 1, 0).

We may also write (22) in the form

$$(25)\qquad \begin{aligned} x_1 &= x_1'x_4'K_2', & x_2 &= x_2'x_4'K_2', \\ x_3 &= x_3'x_4'K_2', & x_4 &= K_4'. \end{aligned}$$

This is also a monoidal transformation but of degree four, the point

* Doehlemann, Geometrischen Transformationen, Band II, Art. 167.

† Sturm, Geo., Verwandtschaften, Band IV, p. 341.

(0, 0, 0, 1) being a triple point. Its image is $x_4'K_2' = 0$. The fundamental curves are of degree $4^2 - 4 = 12$ and consist of the eight lines $K_2' = 0$, $K_4' = 0$ and the plane quartic $x_4' = 0$, $K_4' = 0$. T is a similar transformation. Forming the product ST we find a Cremona transformation of degree 13 for which (0, 0, 0, 1) is a triple point and (0, 0, 1, 0) a nine fold point. The sum of the degrees of the fundamental curves is $13^2 - 13 = 156$. The curves are (*a*) the eight quartics into which T sends the eight lines $K_2' = 0$, $K_4' = 0$; (*b*) the fundamental curves of T counted nine times; (*c*) the three lines $x_3''K_2'' = 0$, $x_4'' = 0$; (*d*) the line $x_1 = 0$, $x_2 = 0$ counted three times.

A plane section of (20) is transformed into a variable curve of degree 36 as before, together with the fixed curves, (*a*) $x_4 = 0$ and (*b*) $x_3 = 0$ counted three times, thus making the total degree 52.

We may also consider ST as the product of a cubic and quartic transformation and thus find similar results. These transformations all have the same meaning for points on the quartic surface, but are distinct for other points.

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